Discrete valuation rings (DVRs)

Before we can go into more depth about the geometry of a curve near a point, we need some more algebra.

Prop/Def: Let R be an integral domain that is not a field. Then R is a DVR if it satisfies either of the two equivalent properties:

1.) R is Noetherian and local and its maximal ideal is principal.

2.) There is an irreducible $t \in R$ s.t. every nonzero $z \in R$ can be written uniquely as $z = ut^n$, u a unit, $n \in \mathbb{Z}_{\geq 0}$.

Pf: 1.) $\Rightarrow$ 2.): Let $m = (t)$ be the maximal ideal.

Suppose $ut^n = vt^m$, u and v units and $n \geq m$.

Then $ut^{n-m} = v$, so t^{n-m} is a unit, so $n = m$, $u = v$. This proves uniqueness.

Now let $z \in R$ be nonzero. If z is a unit, $z = zt^0$, and we're done.

Otherwise, $z \in (t)$, so $z = z_1 t$. If z_1 is a unit, we're done.

Otherwise $z_1 = z_2 t$.

Continuing this process, we get $z_1, z_2, \ldots$ w/ $z_n = z_{n+1} t$.

If some z_i is a unit then $z = z_i t^i$, and we're done.

Otherwise $(z_1) \subseteq (z_2) \subseteq \ldots$, which stabilizes since R is Noetherian.

$\Rightarrow$ $(z_n) = (z_{n+1})$ some n, so $z_{n+1} = v z_n$, some $v \in R$.

So $z_n = z_{n+1} t = v z_n t \Rightarrow z_n (1 - vt) = 0 \Rightarrow vt = 1$, but t isn't a unit, so we get a contradiction.

<u>2.) $\Rightarrow$ 1.)</u>: By 2.), $\mathfrak{m} = (t)$ is exactly the set of nonunits, so R is local, and the max'l ideal is principal.

Thus, we just need to show R is Noetherian. Suppose $I \subseteq R$ is an ideal. Let $n \geq 0$ be the minimum integer s.t. $t^n \in I$. Then $(t^n) \subseteq I$, and if $z \in I$, $z = u t^m \Rightarrow t^m \in I$ so $m \geq n$, so $z \in (t^n)$. Thus $I = (t^n)$. $\square$

<u>Def:</u> t in 2.) is called a <u>uniformizing parameter</u> for R. It is fixed up to multiplication by a unit.

Remark: From the proof, we see that if R is a DVR w/ uniformizing parameter t, then the ideals are exactly $(1) \supset (t) \supset (t^3) \supset \dots$

Ex: Let $a \in \mathbb{A}^1$. Then $\mathcal{O}_a(\mathbb{A}^1) = \left\{ \frac{f}{g} \;\middle|\; \frac{f}{g} \in k(\mathbb{A}^1),\ g(a) \neq 0 \right\}$

The maximal ideal of $\mathcal{O}_a(\mathbb{A}^1) = \{\text{non-units}\} = (x-a)$.

So $\mathcal{O}_a(\mathbb{A}^1)$ is a DVR w/ uniformizing parameter $x-a$.

Ex: The nonunits in $\mathcal{O}_{(0,0)}(\mathbb{A}^2)$ are of the form $\frac{f}{g}$, where

$f(0,0) = 0$. i.e. f has 0 constant term. Thus, $m_{(0,0)}(\mathbb{A}^2) = (x,y)$

But (x,y) is not principal, so $\mathcal{O}_{(0,0)}(\mathbb{A}^2)$ is not a DVR, even though it's local.

Order functions

Let R be a DVR, and fix a uniformizing parameter t.

Let K be the field of fractions of R.

Then for $\frac{f}{g} \in K$, $f = ut^n$, $g = vt^m$, $\frac{f}{g} = \frac{u}{v} t^{n-m}$, $\frac{u}{v} \in R$ a unit.

In fact (exercise) every element $z \in K$ has a <u>unique</u> expression ut^n, u a unit in R, $n \in \mathbb{Z}$.

Def: n is the <u>order</u> of z, denoted $\operatorname{ord}(z)$.

Define $\mathrm{ord}(0) = \infty$. (Exer: ord is independent of the choice of uniformizing parameter)

Then $R = \{z \in K \mid \mathrm{ord}(z) \geq 0\}$, and the max'l ideal is

$\mathfrak{m} = \{z \in K \mid \mathrm{ord}(z) > 0\} \subseteq R$.

Exer: $\mathrm{ord}(ab) = \mathrm{ord}(a) + \mathrm{ord}(b)$

and $\mathrm{ord}(a+b) \geq \min(\mathrm{ord}(a), \mathrm{ord}(b))$.

Quotients of DVRs

Ex: Let $R = \mathcal{O}_0(\mathbb{A}^1)$. Max'l ideal $= \mathfrak{m} = (x)$.

Let $M = (x^n)/(x^{n+1}) \subseteq R/(x^{n+1})$. This is a k-vector space since $k \subseteq R$.

Then every $z \in M$ can be written as $\frac{x^n}{f(x)}$, $f(0) \neq 0$.

Note that $f(x)x^n = f(0)x^n$, since higher powers of x vanish.

$\Rightarrow z = \frac{x^n}{f(x)} = \underbrace{\left(\frac{1}{f(0)}\right)}_{\in k} x^n$ so M is 1-dimensional!

More generally:

Let R be a DVR containing a field k s.t. the composition

$$\begin{tikzcd} k \arrow[r] \arrow[rr, bend right, "\alpha"'] & R \arrow[r] & R/m \end{tikzcd}$$

is an isomorphism.

Let $t \in R$ be a uniformizing parameter. Consider $z \in m^n$.

Then $z = ut^n$, u a unit.

Then the image of u in R/m is nonzero, so $\exists\ \lambda \in k$ s.t. $\alpha(\lambda) = \alpha(u)$.

Thus $u = \lambda + at \in R$, some $a \in R$.

$\Rightarrow z = \lambda t^n + a t^{n+1}$.

But then if we look at $\bar{z} \in m^n/m^{n+1}$, we get $\bar{z} = \lambda \bar{t}^n$.
Thus, $\dim_k (m^n/m^{n+1}) = 1$.

Since $\dim_k(R/m) = 1$, by induction, we get the following:

$$\overbrace{0 \to m^n/m^{n+1} \to R/m^{n+1} \to R/m^n \to 0}^{\text{short exact sequence of } k\text{-vector spaces}}$$

$$\underset{\dim 1}{\uparrow} \qquad \underset{\dim n+1}{\uparrow} \qquad \underset{\dim n}{\uparrow}$$

Note: $\operatorname{ord}(z) = n \Leftrightarrow (z) = m^n$, so $\operatorname{ord}(z) = \dim_k (R/(z))$.

Multiplicities, revisited

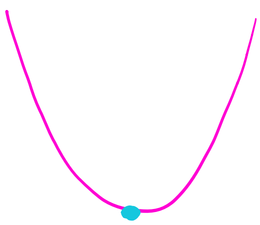
Let f be an irreducible curve. Denote:

$\Gamma(f) := \Gamma(V(f))$, $\mathcal{O}_P(f) := \mathcal{O}_P(V(f))$, $k(f) := k(V(f))$.

Question: If P is a point on f, when is $\mathcal{O}_P(f)$ a DVR?

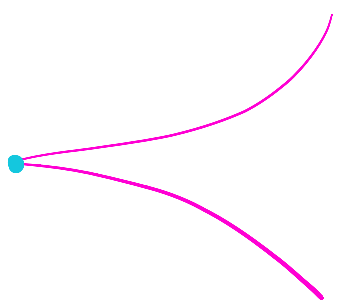
Ex: $f = y - x^2$. $P = (0,0)$.

Then $\mathcal{O}_P(f) = \left\{ \frac{h}{g} \;\middle|\; g(0,0) \neq 0 \right\} \subseteq k(f)$.



$m_P(f) = (x,y) = (x, x^2) = (x)$, so $\mathcal{O}_P(f)$ is a DVR.

Ex: $f = x^3 - y^2$, $P = (0,0)$. In this case the maximal ideal is $m_P(f) = (x,y)$, which is <u>not</u> principal (exercise)



$\Rightarrow \mathcal{O}_P(f)$ is not a DVR.

Thm: Let P be a point on an irreducible curve f. Then for $n >> 0$,

$$\underset{\text{multiplicity}}{m_P(f)} = \dim_k \left(m_P^n / m_P^{n+1} \right)$$

$\nwarrow$ max'l ideal in $\mathcal{O}_P(f)$

Pf sketch: Consider the short exact sequence

$$0 \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to \mathcal{O}_P/\mathfrak{m}^{n+1} \to \mathcal{O}_P/\mathfrak{m}^n \to 0$$

Claim: $\dim_k(\mathcal{O}/\mathfrak{m}^n) = n\, m_P(f) + s$, s a constant, as long as $n \geq m_P(f)$.
(see Thm 2 in section 3.2 of Fulton for pf.)

Then for $n >> 0$, $\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = (n+1)m_P(f) + s - n m_P(f) - s = m_P(f)$. $\square$

So if $\mathcal{O}_P(f)$ is a DVR, $m_P(f) = \dim(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 1$, so P is a simple point. In fact the converse holds:

Thm: Let f be an irreducible plane curve, $P \in V(f)$. Then

P is a simple point of f $\Longleftrightarrow$ $\mathcal{O}_P(f)$ is a DVR.

Pf: ($\Leftarrow$) is done.

($\Rightarrow$) Assume P is a simple point. By changing coordinates, we can assume $P = (0,0)$ and y is the tangent line.

Then x is not tangent to f at P.

We want to show that the maximal ideal $\mathfrak{m} := \mathfrak{m}_P(f) = (x)$.
Note that $\mathfrak{m} = (x, y)$.

$f = y$ + higher deg. terms, so we can write

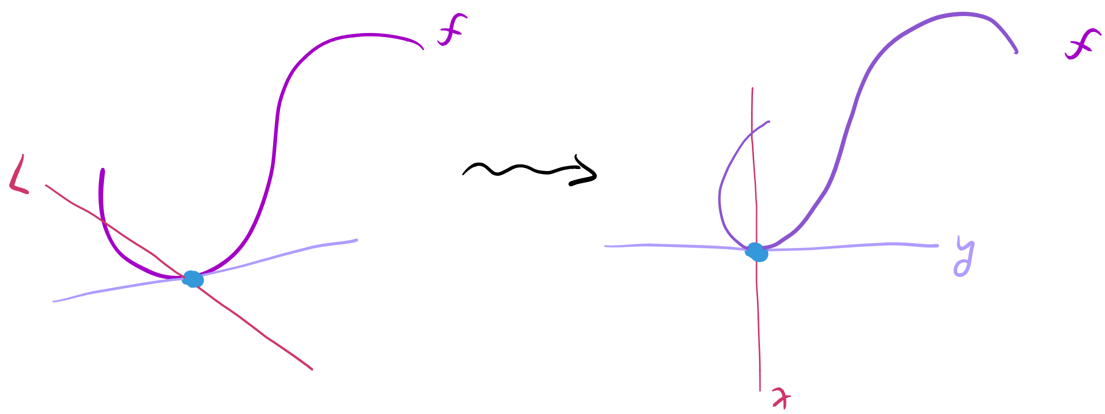
$f = yg - x^2 h$, where $g = 1$ + higher terms, $h \in k[x]$

So in $\Gamma(f)$, $yg = x^2 h$.

$g(0,0) = 1 \neq 0$, so g is a unit in $\mathcal{O}_P(f)$.

Thus, in $\mathcal{O}_P(f)$, $y = \left(\frac{h}{g}\right)x^2 \in (x) \Rightarrow m_P(f) = (x)$. $\square$

In fact, if $L = ax+by+c$ is any line through P not tangent to f at P, there is some change of coordinates s.t. $L \rightsquigarrow x$, and the tangent line $\rightsquigarrow y$. Thus, L is a uniformizing parameter in $\mathcal{O}_P(f)$.



The order function at simple points

Let P be a simple point on an irreducible curve f.

Then $\mathcal{O}_P(f)$ is a DVR, so there is an order function ord_P^f on the field of fractions of $\mathcal{O}_P(f)$, which is equal to $k(f)$.

$$\left(\text{Recall } \Gamma(f) \subseteq \mathcal{O}_P(f) \subseteq k(f)\right)$$

Let L be a line through P.

If L is not tangent to f at P, then $L \in \mathcal{O}_P(f)$ is a uniformizing parameter, so $\mathrm{ord}_P^f(L) = 1$.

If L <u>is</u> tangent to f at P, we can assume we're in the situation where $L = y$ and $y = \frac{x^2 h}{g}$.

$g(P) \neq 0$, so x doesn't divide g, while x <u>may</u> divide h, so $\mathrm{ord}_P^f(L) \geq 2$.

To summarize...

Thm: Let P be a simple point on an irreducible curve f. Let L be a line.

1.) $\mathrm{ord}_P^f(L) = 0 \iff L$ doesn't contain P

2.) $\mathrm{ord}_P^f(L) = 1 \iff L$ passes through P but is not tangent to f at P.

3.) $\mathrm{ord}_P^f(L) \geq 2 \iff L$ is tangent to f at P.